2020

MATHEMATICS (Honours)

Paper : MTMH - DC- 05

[CBCS]

Full Marks : 32 Time : Two Hours

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meanings.

Group - A

1. Answer any **four** questions. $1 \times 4 = 4$

(a) State the Dirichlet conditions for convergence of a Fourier series.

(b) Determine whether the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 2 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ 3 & \text{if } x \in \mathbb{Q} \end{cases}$$

is R-integrable on $[0, 1]$.

(c) Test the convergence of the improper integral

$$\int_0^1 \frac{1}{x^2} dx.$$

(d) Prove that $\int_0^\infty t^\alpha e^{-t^2} dt = \frac{1}{2}\Gamma\left(\frac{\alpha+1}{2}\right)$, $\alpha > 1$.

(e) For a periodic function f of period 2π, prove that

$\int_{\alpha}^{\beta} f(x)dx = \int_{\alpha+2\pi}^{\beta+2\pi} f(x)dx.$

(f) Compute the total variation $V_f[0,3]$ for the function

$f(x) = x^2 - 4x + 3$ on $[0,3]$.

(g) Find the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$.

Group - B

Answer any *two* questions. 5×2=10

2. Let $a, b \in \mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$. Show that f is bounded on $[a,b]$. [5]

3. State and prove the fundamental theorem of integral calculus. [1+4]

4. Find the Fourier series expansion of

$$f(x) = \begin{cases} \cos x & \text{if } 0 \leq x \leq \pi \\ -\cos x & \text{if } -\pi \leq x < 0. \end{cases}$$ [5]

5. Test the uniform convergence of the sequence $\{f_n\}$ on $[0,1]$, where

$f_n(x) = x^{n-1}(1-x)$. [5]

Group - C

Answer any *two* questions. 9×2=18

6. (a) Let $\{f_n\}$ be a sequence of functions defined on a subset X of $\mathbb{R}$ and let $\lim_{x\to\infty} f_n(x) = f(x), \ \forall\, x \in X$ and $M_n = \sup\{|f_n(x) - f(x)| : x \in X\}$. Show that $\{f_n\}$ converges uniformly if and only if $M_n \to 0$ and $n \to \infty$. [4]

   (b) Test the convergence of
$$\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}.$$ [5]

7. (a) If f is bounded and integrable on $[a,b]$, then show that $|f|$ is also bounded and integrable on $[a,b]$. Also show that
$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|\,dx.$$ [2+3]

   (b) Let $f : [0,1] \to \mathbb{R}$ be defined by
$$f(x) = \begin{cases} x^2 \cos\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$
Show that f is a function of bounded variation on $[0,1]$. [4]

8. (a) Let f be a periodic function with period 2π which is bounded and integrable on $(-\pi, \pi)$. If $(-\pi, \pi)$ is divided into a finite number of open sub-intervals in each of which f is monotonic, then show that

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nc + b_n \sin nc)$$
$$= \begin{cases} \frac{1}{2}\left[f(c-0) + f(c+0)\right] & \text{for } -\pi < c < \pi \\ \frac{1}{2}\left[f(\pi-0) + f(-\pi+0)\right] & \text{for } c = \pm\pi. \end{cases}$$

[5]

(b) From the expression

$$\tan^{-1} x = \int_0^x \frac{dx}{1+x^2},$$

obtain the series

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots .$$

Find the range of x for which the preceding expression holds. Also deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots .$$

[1+1+2]