2020

MATHEMATICS (Honours)

Paper : MTMH - DC- 06

[CBCS]

Full Marks : 32 Time : Two Hours

The figures in the margin indicate full marks.
Candidates are required to give their answers
in their own words as far as practicable.

Notations and symbols have their usual meanings.

Group - A

1. Answer any **four** questions. $1 \times 4 = 4$

(a) Find the values of k for which the set $\{(k, 6), (2, k)\}$ forms a basis for $\mathbb{R}^2$.

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by $T(x, y) = (x, 0)$. Find $\ker T$.

(c) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by $T(x, y) = (y, x)$. Find the dimension of range of T.

(d) Give an example of a linear operator T on $\mathbb{R}^2$ such that T has no eigenvalues.

(e) Let T be a linear operator on $\mathbb{P}_2(\mathbb{R})$ defined by $T(f(x)) = f'(x)$. Find the characteristic polynomial of T.

(f) Prove that in an inner product space V, $\|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$ for all $x, y \in V$.

(g) Let T and U be self-adjoint operators on a finite dimensional inner product space V. If TU is self-adjoint, then prove that $TU = UT$.

Group - B

Answer any *two* questions. 5×2=10

2. (a) Prove that a nonempty subset S of a vector space V over the field F is a subspace of V if and only if the following condition holds.

   (i) $x + y \in S, \forall\, x, y \in S$.

   (ii) $\lambda x \in S, \forall\, x \in S$ and $\lambda \in F$. [3]

   (b) Let $\boldsymbol{B}$ be a basis for the finite dimensional inner product space V. If $x \in V$ is such that $\langle x, z \rangle = 0$, $\forall z \in \boldsymbol{B}$, then prove that $x = \theta$. [2]

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator given by $T(x, y, z) = (0, 0, x)$. Find the matrix representation of T with respect to the standard ordered basis for $\mathbb{R}^3$. Also determine whether T is diagonalizable or not. [2+3]

4. Let $S = \{(1, 1, 1), (0, 1, 1), (0, 0, 1)\}$ be a subset of the inner product space $\mathbb{R}^3$. Apply Gram-Schmidt orthogonalization process on S to obtain an orthonormal basis for span(S). [5]

5. Let $M_2(\mathbb{R})$ be the set of all 2×2 real matrices and $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) : a + b = 0 \right\}$ and $T = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) : c + d = 0 \right\}$. Find the dimension of S, T and $S \cap T$. [5]

Group - C

Answer any *two* questions. 9×2=18

6. (a) Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by $T(x, y, z, t) = (x - y + z + t, x + 2z - t, x + y + 3z - 3t)$. Find a basis for

   (i) the range of T

   (ii) the kernel of T.

   Also find their dimensions. [2+2+1]

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by $T(x, y, z) = (2x + 3y + 5z, 4x - 5y + 6z, 5x + 7y + 2z)$. Find the matrix representation of T relative to the basis $S = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$. [4]

7. (a) For the subsets $A = \{x, y, z\}$ and $B = \{x + y, y, y + z, x + z\}$ of the vector space $\mathbb{R}^3$, prove that $\text{span}(A) = \text{span}(B)$. [3]

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator given by $T(x, y) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$, where $0 < \theta < \pi$. Determine whether T is

(i) self-adjoint

(ii) unitary

(iii) normal. [2+2+2]

8. (a) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(x, y) = (3x + y, x + 3y)$. Determine whether the vectors $v_1 = (1, -1)$ and $v_2 = (1, 1)$ are eigenvectors of T. [2]

(b) In an inner product space V, prove that $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$ for all $x, y \in V$. [5]

(c) Prove that every eigenvalue of a self-adjoint operator T on a finite dimensional inner product space V is real. [2]