U.G. 3rd Semester Examination 2021

MATHEMATICS (Honours)

Paper : DC-7

[Multivariate Calculus and Vector Calculus]

(CBCS)

Full Marks : 32 Time : 2 Hours

The figures in the margin indicate full marks.
Notations and symbols have their usual meanings.

Group - A

(4 Marks)

1. Answer any ***four*** questions : 1×4=4

(a) Show that $\lim\limits_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4}$ does not exist.

(b) If $u = tan^{-1}\frac{x^3+y^3}{x-y}$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = sin\,2u$.

(c) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = x^3 + y^3 - 3x - 12y + 20$. Find the stationary points of this function.

(d) Change the order of integration in $\int_0^1 dx \int_x^{\sqrt{x}} f(x,y)\,dy$.

(e) Evaluate $\iint_R \sin(x+y)\,dx\,dy$ where $R = \left[0, \frac{\pi}{2}; 0, \frac{\pi}{2}\right]$.

(f) Let C be the boundary of the region $R=\{(x, y) \in \mathbb{R}^2 : -1 \le y \le 1, 0 \le x \le 1 - y^2\}$ oriented in the counter clockwise direction. Then find the value of $\int_C y\,dx + 2x\,dy$.

(g) Let $f(x,y)=\begin{cases}\dfrac{|x|}{|x|+|y|}\sqrt{x^4+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$.

Find f_x (0, 0) and f_y (0, 0).

Group - B

(10 Marks)

Answer any ***two*** questions : 2×5=10

2. (a) The function f, defined over the whole xy-plane, is given by

$$f(x,y)=\begin{cases}\dfrac{|x|}{y^2}e^{-|x|/y^2}, & y\neq 0\\ 0, & y=0\end{cases}$$

Discuss the existence of the limit as $(x,y)\to(0,0)$. 3

(b) Evaluate $\iint_R [x+y]\,dx\,dy$, over R = [0, 1; 0, 2], where [x + y] denotes the greatest integer less than or equal to ($x + y$). 2

3. Let $f(x,y)=\begin{cases}x^2\sin\dfrac{1}{x}+y^2\sin\dfrac{1}{y}, & \text{when neither } x=0 \text{ nor } y=0;\\ x^2\sin\dfrac{1}{x}, & \text{when } x\neq 0,\ y=0;\\ y^2\sin\dfrac{1}{y}, & \text{when } y\neq 0,\ x=0;\\ 0, & \text{when } x=0,\ y=0.\end{cases}$

Show that $f_x(x,y)$ and $f_y(x,y)$ are discontinuous at (0, 0) but $f(x, y)$ is differentiable at (0, 0). 5

4. Show that $\int_0^1 dx\int_0^{\sqrt{1-x^2}}\dfrac{dy}{(1+e^y)\sqrt{1-x^2-y^2}}=\dfrac{\pi}{2}\log\dfrac{2e}{1+e}$, by changing the order of integration. 5

5. Evaluate the line integral of $F = \sin y i + x(1 + cos\, y) j$ over the circular path given by $x^2 + y^2 = a^2, z = 0$. 5

Group - C

(18 Marks)

Answer any ***two*** questions : 2×9=18

6. (a) If $z = xf(x+y) + yg(x+y)$, prove that $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$. 5

(b) Show that for 0 < θ < 1,

$$\sin x \sin y = xy - \frac{1}{6}\left[\left(x^3 + 3xy^2\right)\cos\theta x \sin\theta y + \left(y^3 + 3x^2 y\right)\sin\theta x \cos\theta y\right].$$ 4

7. (a) If $xyz = a^2(x+y+z)$, using Lagrange's method of multipliers, show that the minimum value of $yz + zx + xy$ is $9a^2$. 4

(b) If R be the interior of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, show that

$$\iiint_R \sqrt{a^2b^2c^2 - b^2c^2x^2 - c^2a^2y^2 - a^2b^2z^2}\ dxdydz = \frac{1}{4}\pi^2 a^2 b^2 c^2$$ 5

8. (a) Using Stoke's Theorem, show that

$$\iint_S (y-z)\,dy\,dz + (z-x)\,dz\,dx + (x-y)\,dx\,dy = \pi a^3$$

Where S is the portion of the surface $x^2 + y^2 - 2ax + az = 0,\ z \geq 0$. 5

(b) Show that $\int_0^\pi \int_0^\pi \left|\cos(x+y)\right| dx\,dy = 2\pi$ by the substitution $x = u - v, y = v$. 4