Coulomb's Law, Principle of Superposition, Electrostatic Field and Field Lines

Electric charge is a fundamental property of matter responsible for electrical interactions.

There are two types of charge: **Positive charge (+)** & **Negative charge (−)**

Basic properties of charge

1. **Like charges repel** each other. **Unlike charges attract** each other.
2. Charge is **conserved**. Charge is **quantized**:

$$q = ne$$

where n is an integer and $e = 1.602 \times 10^{-19}\ \mathrm{C}$

The SI unit of charge is the **coulomb (C)**.

Coulomb's Law

Statement:
The electrostatic force between two stationary point charges is directly proportional to the product of their charges and inversely proportional to the square of the distance between them.

Let two point charges q_1 and q_2 be separated by a distance r.

Then, $\boxed{F \propto \frac{q_1 q_2}{r^2}}$

Therefore, $\boxed{F = k\frac{|q_1 q_2|}{r^2}}$ where $k = \frac{1}{4\pi\varepsilon_0}$

Hence, in vacuum, $\boxed{F = \frac{1}{4\pi\varepsilon_0}\frac{|q_1 q_2|}{r^2}}$

where ε_0 is the permittivity of free space and $\varepsilon_0 = 8.854 \times 10^{-12}\ \mathrm{C^2N^{-1}m^{-2}}$.

$$\boxed{k = 9 \times 10^9\ \mathrm{Nm^2C^{-2}}}$$

Vector form: The force on q_2 due to q_1 is $\boxed{\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{r}_{12}}$

The force acts along the line joining the two charges.

Principle of Superposition

In many physical situations, more than two charges are present

Statement:
The total electrostatic force acting on a charge due to a number of other charges is equal to the vector sum of the individual forces produced by each charge separately.

Suppose a test charge q is surrounded by charges $q_1, q_2, \ldots, q_n$. Then total force on q is

$$\boxed{\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots + \vec{F}_n} \quad \text{or,} \quad \boxed{\vec{F} = \sum_{i=1}^{n} \vec{F}_i}$$

For each source charge, $\vec{F}_i = \frac{1}{4\pi\varepsilon_0}\frac{q q_i}{r_i^2}\hat{r}_i$ For point charges, $\boxed{\vec{F} = \frac{q}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{r}_i}$

Thus, the forces due to individual charges are calculated separately and then added vectorially.

For distribution of charges:

Electrostatic Field

Coulomb's law describes the force between charges. However, it is often more useful to describe the effect produced by a source charge in the surrounding space. This leads to the concept of the electric field.

Consider a source charge Q. Place a small positive test charge $q_0$at a point near it. If the test charge experiences a force $\vec{F}$, the electric field at that point is defined as $\boxed{\vec{E} = \lim_{q_0 \to 0} \frac{\vec{F}}{q_0}}$

In elementary applications, this is commonly written as $\boxed{\vec{E} = \frac{\vec{F}}{q_0}}$

The test charge is taken sufficiently small so that it does not significantly disturb the existing charge distribution.

Definition:

The electrostatic field at a point is defined as the force experienced by a unit positive test charge placed at that point.

The SI unit of electric field is $\boxed{\text{N/C}}$ *or* $\boxed{\text{V/m}}$

Dimensional formula $\boxed{[MLT^{-3}I^{-1}]}$

Electric Field Due to a Point Charge

Consider a point charge $Q$and a test charge $q_0$placed at a distance r.

According to Coulomb's law, $F = \frac{1}{4\pi\varepsilon_0}\frac{|Qq_0|}{r^2}$

Since $E = \frac{F}{q_0}$, we get $\boxed{E = \frac{1}{4\pi\varepsilon_0}\frac{|Q|}{r^2}}$ **Vector form** $\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}}$

Direction of electric field

For a **positive charge**, the field points **radially outward**.

For a **negative charge**, the field points **radially inward**.

Electric Field Due to a System of Charges

According to the principle of superposition, the resultant electric field due to several charges is the vector sum of the fields produced by the individual charges.

Thus, $\boxed{\vec{E} = \vec{E}_1 + \vec{E}_2 + \cdots + \vec{E}_n}$ or, $\boxed{\vec{E} = \sum_{i=1}^{n} \vec{E}_i}$

For n point charges, $\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n} \frac{q_i}{r_i^2}\hat{r}_i}$

Force on a Charge in an Electric Field

If a charge $q$is placed in an electric field $\vec{E}$, the force acting on it is $\boxed{\vec{F} = q\vec{E}}$

For a positive charge, the force is in the direction of the electric field.

For a negative charge, the force is opposite to the direction of the electric field.

Electric Field Lines:
An electric field line is an imaginary curve drawn in an electric field such that the tangent to the curve at any point gives the direction of the electric field at that point.
Field lines are also called **lines of force**.

Properties of Electric Field Lines

1. Electric field lines originate from **positive charges** and terminate on **negative charges**.
2. For an isolated positive charge, field lines are directed **outward**.
3. For an isolated negative charge, field lines are directed **inward**.
4. The tangent to an electric field line at any point gives the direction of the electric field at that point.
5. Two electric field lines **never intersect** each other.
6. The density of field lines represents the strength of the electric field.

   Closely spaced lines → stronger field. Widely spaced lines → weaker field.
7. Electric field lines are perpendicular to the surface of a conductor in electrostatic equilibrium.
8. Electric field lines do not form closed loops in electrostatics.
9. The number of field lines associated with a charge is proportional to the magnitude of the charge.

Uniform Electric Field

An electric field is said to be **uniform** when its magnitude and direction remain the same at every point. The field lines of a uniform electric field are:

- straight,
- parallel, and
- equally spaced.
- For example, the electric field between two large, oppositely charged parallel plates is approximately uniform away from the edges.

Thus, $\boxed{\vec{E} = \text{constant}}$

Important Formulae for Examination

Quantity	Formula
Coulomb force	$F = k\frac{\mid q_1 q_2 \mid}{r^2}$
Electric field	$E = \frac{F}{q_0}$
Force in electric field	$\vec{F} = q\vec{E}$
Superposition of forces	$\vec{F} = \sum_i \vec{F}_i$
Superposition of fields	$\vec{E} = \sum_i \vec{E}_i$
Coulomb constant	$k = \frac{1}{4\pi\varepsilon_0} \approx 9 \times 10^9$

Very Short Exam Answers

Q. What is Coulomb's law?
Coulomb's law states that the electrostatic force between two point charges is directly proportional to the product of their charges and inversely proportional to the square of the distance between them.

Q. Define electric field.
Electric field at a point is the force experienced by a unit positive test charge placed at that point.

$$\boxed{\vec{E} = \frac{\vec{F}}{q_0}}$$

Q. What is the principle of superposition?
The total electrostatic force or electric field due to several charges is the vector sum of the forces or fields produced by the individual charges.

Q. What are electric field lines?
Electric field lines are imaginary curves whose tangent at any point gives the direction of the electric field at that point.

Q. Why do electric field lines never intersect?
Because the electric field at a point has a unique direction. If two field lines intersected, there would be two directions of electric field at the same point, which is impossible.

Mathematical problems:

Electric Flux

Electric flux is a measure of the **electric field passing through a surface**.
Imagine the electric field lines as a fluid flowing through space. The electric flux is the total amount of this "fluid" passing through a specific area. It helps us quantify how much the electric field "pierces" a surface.

Consider a small area element $d\vec{A}$ in an electric field $\vec{E}$. The electric flux through the area element is $d\Phi_E = \vec{E} \cdot d\vec{A}$

Therefore, $\boxed{d\Phi_E = E\, dA \cos\theta}$

where θ= angle between $\vec{E}$ and the normal to the surface.

For a finite surface, $\boxed{\Phi_E = \int_S \vec{E} \cdot d\vec{A}}$

Uniform Electric Field

If the electric field is uniform and the surface is plane, $\boxed{\Phi_E = EA\cos\theta}$

Special Cases (i) When $\theta = 0°$: $\Phi_E = EA$ Flux is maximum.

(ii) When $\theta = 90°$: $\Phi_E = 0$ There is no flux through the surface.

(iii) When $\theta = 180°$: $\Phi_E = -EA$
The negative sign indicates that the electric field is opposite to the outward normal.

SI Unit of Electric Flux $\boxed{\mathrm{N\,m^2/C}}$ which is also equivalent to Volt-meters (V·m).

Electric Flux Through a Closed Surface

For a closed surface, the area vector $d\vec{A}$ is always directed **outward**.

Hence, the total electric flux through a closed surface is $\boxed{\Phi_E = \oint_S \vec{E} \cdot d\vec{A}}$

If electric field lines come out of the closed surface, the flux is positive.

If electric field lines enter the closed surface, the flux is negative.

Divergence of a Vector Field

The **divergence** of a vector field represents the net outward flux per unit volume at a point. For an electric field $\vec{E}$, $\boxed{\nabla \cdot \vec{E} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint_S \vec{E} \cdot d\vec{A}}$ where, $\boxed{\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}}$

Thus, divergence tells us whether the electric field is acting as a **source or sink** at a particular point.

Positive Divergence: The point is a **source** of electric field lines. More field lines are leaving an infinitesimal volume around the point than entering it. This occurs at the location of positive electric charges.

Negative Divergence: The point is a **sink** of electric field lines. More field lines are entering the volume than leaving it. This occurs at the location of negative electric charges.

Zero Divergence: The number of field lines entering the volume equals the number of field lines leaving. This is true for any point in empty space (a charge-free region).

Gauss's Theorem of Electrostatics

The total electric flux through any closed surface is equal to the net charge enclosed by that surface divided by the permittivity of free space (ε_0). $\boxed{\oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enc}}}{\varepsilon_0}}$

If the charge density is ρ, then $Q_{\text{enc}} = \int_V \rho \ dV$

Therefore, $\oint_S \vec{E} \cdot d\vec{A} = \frac{1}{\varepsilon_0} \int_V \rho \ dV$

Using the divergence theorem, $\oint_S \vec{E} \cdot d\vec{A} = \int_V (\nabla \cdot \vec{E}) \, dV$

Hence, $\int_V (\nabla \cdot \vec{E}) \, dV = \frac{1}{\varepsilon_0} \int_V \rho \ dV$

Since this relation is true for any volume, $\boxed{\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}}$

This is the **differential form of Gauss's law**.

In a charge-free region $\rho = 0$ Then $\boxed{\nabla \cdot \vec{E} = 0}$

Thus, the divergence of an electrostatic field is directly related to the volume charge density.

Proof of Gauss's Theorem for a Point Charge

Consider a point charge $q$placed at the centre of a spherical surface of radius r.
The electric field at any point on the sphere is $E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2}$
Since the electric field is radial, it is parallel to the area vector $d\vec{A}$.

Therefore, $\vec{E} \cdot d\vec{A} = E \, dA$ The total flux is $\Phi_E = \oint_S E \ dA$
Since E is constant over the spherical surface, $\Phi_E = E \oint_S d\,A$
The surface area of a sphere is $4\pi r^2$
Hence, $\Phi_E = \frac{q}{4\pi\varepsilon_0 r^2} (4\pi r^2)$ Therefore, $\boxed{\Phi_E = \frac{q}{\varepsilon_0}}$

Thus, $\boxed{\oint_S \vec{E} \cdot d\vec{A} = \frac{q}{\varepsilon_0}}$

Hence, Gauss's theorem is proved for a point charge.

8. Important Features of Gauss's Theorem

1. Gauss's theorem applies to **any closed surface**.
2. The net electric flux depends only on the **total charge enclosed** by the surface.
3. Charges outside the closed surface produce zero **net flux** through it.
4. The shape and size of the Gaussian surface do not affect the total flux, provided the enclosed charge remains the same.
5. Gauss's law is particularly useful for calculating electric fields when the charge distribution possesses a high degree of symmetry.

Common Symmetries and Gaussian Surfaces:
A **Gaussian surface** is an imaginary closed surface chosen to apply Gauss's theorem conveniently. The shape of the Gaussian surface is selected according to the symmetry of the charge distribution.

1. **Spherical Symmetry:** For a point charge or a uniformly charged sphere, choose a spherical Gaussian surface centered on the charge distribution. E is radial and constant in magnitude on the surface.
2. **Cylindrical Symmetry:** For an infinite line charge or an infinitely long charged cylinder, choose a cylindrical Gaussian surface coaxial with the charge distribution. E is radial and constant in magnitude on the curved surface.
3. **Planar Symmetry:** For an infinite sheet of charge, choose a "pillbox" (a small cylinder) as the Gaussian surface, with its flat faces parallel to the sheet. E is perpendicular to the sheet and constant in magnitude on the flat faces.

The most important point is that a Gaussian surface is an **imaginary mathematical surface**, not necessarily a physical surface.

Important Properties of Gauss's Theorem

1. Gauss's theorem applies only to a **closed surface**.
2. The total flux depends only on the **net charge enclosed** by the surface.
3. Charges outside the closed surface do not change the **net electric flux** through it.
4. The shape and size of the Gaussian surface do not affect the total flux if the enclosed charge remains the same.
5. Gauss's theorem is particularly useful for charge distributions having high symmetry.

Very Short Examination Answers

What is electric flux?

Electric flux is the measure of the electric field passing through a surface. $\boxed{\Phi_E = \int_S \vec{E} \cdot d\vec{A}}$

What is divergence of an electric field?

The divergence of an electric field at a point is the net outward electric flux per unit volume surrounding that point. $\boxed{\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}}$

State Gauss's theorem.

The total electric flux through any closed surface is equal to the total charge enclosed by the surface divided by the permittivity of free space. $\boxed{\oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enc}}}{\varepsilon_0}}$

What is a Gaussian surface?

A Gaussian surface is an imaginary closed surface selected according to the symmetry of a charge distribution for convenient application of Gauss's law.

Applications of Gauss's Theorem to Determine Electric Field

General Procedure for Applying Gauss's Theorem

Step 1: Identify the symmetry

Determine whether the charge distribution has spherical, cylindrical, or planar symmetry.

Step 2: Choose a Gaussian surface

The Gaussian surface should have the same symmetry as the charge distribution.

Step 3: Determine the electric field

Use symmetry to determine the direction and magnitude of $\vec{E}$on the Gaussian surface.

Step 4: Calculate electric flux

$$\Phi_E = \oint_S \vec{E} \cdot d\vec{A}$$

Step 5: Calculate enclosed charge

Find Q_{enc}inside the Gaussian surface.

Step 6: Apply Gauss's theorem

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enc}}}{\varepsilon_0}$$

Step 7: Obtain *E*: Solve the resulting equation for the electric field.

Electric Field Due to a Spherically Symmetric Charge Distribution:

Spherical symmetry occurs when the charge distribution is symmetric about a point.

Examples include:

- point charge,
- uniformly charged spherical shell,
- uniformly charged solid sphere.

For spherical symmetry, the appropriate Gaussian surface is a **sphere**.

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enc}}}{\varepsilon_0}$$

$$\oint \vec{E} \cdot d\vec{A} = E(4\pi r^2)$$

A. Electric Field Due to a Point Charge

Consider a point charge q placed at the centre of a spherical Gaussian surface of radius r. The electric field at every point on the spherical surface has the same magnitude and is directed radially outward. According to Gauss's theorem,

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{q}{\varepsilon_0}$$

Since $\vec{E}$is parallel to $d\vec{A}$, $\quad \vec{E} \cdot d\vec{A} = E\,dA$

Therefore, $\quad E \oint_S d\,A = \frac{q}{\varepsilon_0}$

The surface area of the sphere is $\quad 4\pi r^2$

Hence, $\quad E(4\pi r^2) = \frac{q}{\varepsilon_0}$

Therefore, $\quad \boxed{E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}}$

Thus, the electric field due to a point charge varies as $\boxed{E \propto \frac{1}{r^2}}$ and is directed radially outward for a positive and radially inward for a negative charge.

Electric Field Due to a Uniformly Charged Spherical Shell

Consider a thin spherical shell of radius R carrying a total charge Q uniformly distributed over its surface. We consider two cases.

Case I: Point Outside the Spherical Shell $(r > R)$

Choose a spherical Gaussian surface of radius r, where $r > R$. The entire charge Q is enclosed.

By Gauss's theorem, $\quad \oint_S \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0}$

Since $E$is constant over the Gaussian surface, $E(4\pi r^2) = \frac{Q}{\varepsilon_0}$

Therefore, $\quad \boxed{E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}}$

Thus, outside a uniformly charged spherical shell, the electric field is the same as that produced by a point charge Q placed at its centre.

Case II: Point Inside the Spherical Shell $(r < R)$

Choose a spherical Gaussian surface of radius $r < R$. No charge is enclosed by this surface.

Therefore, $\quad Q_{\text{enc}} = 0$

Using Gauss's theorem, $\quad E(4\pi r^2) = 0$

Hence, $\boxed{E = 0}$

Therefore, the electric field at every point inside a uniformly charged spherical shell is zero.

Result for a Charged Spherical Shell

$$\boxed{E = \begin{cases} 0, & r < R \\ \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2}, & r > R \end{cases}}$$

At the surface, the field just outside is $\boxed{E = \frac{Q}{4\pi\varepsilon_0 R^2}}$

Electric Field Due to a Uniformly Charged Solid Sphere

Consider a solid sphere of radius Rhaving uniform volume charge density ρ.
Again, two cases are considered.

Case I: Point Outside the Sphere ($r > R$)

The entire charge of the sphere is enclosed. Total charge, $Q = \rho\left(\frac{4}{3}\pi R^3\right)$

By Gauss's theorem, $E(4\pi r^2) = \frac{Q}{\varepsilon_0}$

Therefore, $\boxed{E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}}$

Thus, outside a uniformly charged solid sphere, the field is the same as that of a point charge Q at the centre.

Case II: Point Inside the Sphere ($r < R$)

Consider a spherical Gaussian surface of radius r. The charge enclosed is $Q_{\text{enc}} = \rho\left(\frac{4}{3}\pi r^3\right)$

Applying Gauss's theorem, $E(4\pi r^2) = \frac{1}{\varepsilon_0}\rho\left(\frac{4}{3}\pi r^3\right)$

Therefore, $\boxed{E = \frac{\rho r}{3\varepsilon_0}}$

Thus, inside a uniformly charged solid sphere, $\boxed{E \propto r}$

Electric Field Due to an Infinite Line Charge

Consider an infinitely long straight wire having uniform linear charge density $\lambda = \frac{Q}{L}$
We need to find the electric field at a perpendicular distance r from the wire.

Due to cylindrical symmetry, the appropriate Gaussian surface is a **cylinder** of radius r and length L, with the wire along its axis. The electric field is radial and has the same magnitude at every point on the curved surface.

Electric Flux

There is no flux through the two flat circular ends because $\vec{E}$ is parallel to those surfaces. Therefore, flux is only through the curved surface. Curved surface area is $A = 2\pi rL$
Hence, $\Phi_E = E(2\pi rL)$
The charge enclosed is $Q_{\text{enc}} = \lambda L$
Using Gauss's theorem, $E(2\pi rL) = \frac{\lambda L}{\varepsilon_0}$

Therefore, $\boxed{E = \frac{\lambda}{2\pi\varepsilon_0 r}}$

Thus, the electric field due to an infinite line charge varies as $\boxed{E \propto \frac{1}{r}}$

Direction

For a positive line charge, the field is radially outward.
For a negative line charge, the field is radially inward.

Electric Field Due to an Infinite Plane Sheet of Charge

Consider an infinite plane sheet having uniform surface charge density $\sigma = \frac{Q}{A}$
We want to find the electric field near the sheet.
Due to planar symmetry, the electric field is perpendicular to the plane and has the same magnitude at equal distances on both sides.
A suitable Gaussian surface is a short cylindrical **pillbox** passing through the sheet.
Let the area of each circular face be A.

The electric field is perpendicular to the two flat faces. Therefore, flux through the two faces is
$\Phi_E = EA + EA$
or $\Phi_E = 2EA$
There is no flux through the curved surface.

The charge enclosed is $Q_{\text{enc}} = \sigma A$
Applying Gauss's theorem, $2EA = \frac{\sigma A}{\varepsilon_0}$

Therefore, $\boxed{E = \frac{\sigma}{2\varepsilon_0}}$

Thus, the electric field due to an infinite plane sheet of charge is independent of the distance from the sheet. $\boxed{E = \frac{\sigma}{2\varepsilon_0}}$

Electric Field Near a Conducting Plane Surface

For a charged conductor in electrostatic equilibrium, the electric field inside the conductor is zero. Consider a small Gaussian pillbox with one face just inside the conductor and the other face just outside it. Let the surface charge density be σ. Since the field inside the conductor is zero, flux occurs only through the outer face. Thus, $\Phi_E = EA$
The charge enclosed is $Q_{\text{enc}} = \sigma A$
By Gauss's theorem, $EA = \frac{\sigma A}{\varepsilon_0}$

Therefore, $\boxed{E = \frac{\sigma}{\varepsilon_0}}$

Hence, the electric field just outside a charged conductor is $\boxed{E = \frac{\sigma}{\varepsilon_0}}$

and inside the conductor, $\boxed{E = 0}$

Comparison of Important Results

Charge configuration	Gaussian surface	Electric field
Point charge q	Sphere	$E = \frac{q}{4\pi\varepsilon_0 r^2}$
Spherical shell, $r < R$	Sphere	$E = 0$
Spherical shell, $r > R$	Sphere	$E = \frac{Q}{4\pi\varepsilon_0 r^2}$
Uniform solid sphere, $r < R$	Sphere	$E = \frac{\rho r}{3\varepsilon_0}$
Uniform solid sphere, $r > R$	Sphere	$E = \frac{Q}{4\pi\varepsilon_0 r^2}$
Infinite line charge	Cylinder	$E = \frac{\lambda}{2\pi\varepsilon_0 r}$
Infinite plane sheet	Pillbox	$E = \frac{\sigma}{2\varepsilon_0}$
Charged conducting surface	Pillbox	$E = \frac{\sigma}{\varepsilon_0}$

Important Points for Examinations

Spherical symmetry: Use a **spherical Gaussian surface**. $\boxed{E \propto \frac{1}{r^2}}$ for a point charge or outside a spherically symmetric charge distribution.

Cylindrical symmetry: Use a **cylindrical Gaussian surface**. $\boxed{E \propto \frac{1}{r}}$ for an infinite line charge.

Planar symmetry: Use a **pillbox Gaussian surface**. $\boxed{E = \frac{\sigma}{2\varepsilon_0}}$ for an infinite non-conducting plane sheet.

For a conducting surface, $\boxed{E = \frac{\sigma}{\varepsilon_0}}$